The geodesic flow on symmetric spaces

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Tuesday 9th April, 2019

Abstract

The goal of this talk is to understands geodesics and the geodesic flow on "nice" spaces like symmetric spaces. We'll see that on such spaces, the geometry of the space can be translated more or less completely into linear algebra. This will also allow us to give an easy proof of the fact that the geodesic flow on a negatively curved space is ergodic. We'll also work out a lot of examples, where the translation to linear algebra is easy to see, like the hyperbolic plane $\mathbb{H}^2$. This talk will assume no prior knowledge of ergodic theory or symmetric spaces.

Contents

1 The geodesic flow

Consider a compact Riemannian manifold M: a question of central importance is how do the geodesics on the manifold look like. One might ask whether they close up, whether some geodesic visits the neighbourhood of every point (i.e. whether its image is dense), what fraction of time does a generic geodesic spend in some open set, and so on. One can rephrase these questions as questions about an action of $\mathbb{R}$ on the unit tangent bundle T^1M on the manifold M. The action is given by the following formula.

$$\begin{aligned} \phi &: \mathbb{R} \times T^1M \to T^1M \\ \phi &: (t, (p, v)) \mapsto (\exp_p(tv), d(\exp_p(tv))) \end{aligned}$$

The expression for the geodesic flow may look rather formidable, but it's a rather natural thing to consider. If one thinks of the $\mathbb{R}$ coordinate as time, given a point $p \in M$, and a unit tangent vector v, the geodesic flow will send it along a geodesic in direction v for time t. Informally, think of standing at point p and facing direction v: if you walk in a straight line for time t, you'll end up where the geodesic flow sends $(t, (p, v))$.

The geodesic flow gives us a dynamical system we can study in order to understand the geometry of the space M. Coming back to the questions we raised at the beginning, closed geodesics correspond exactly to the periodic points on the geodesic flow, and to show that some geodesic has dense image, it suffices to prove that the corresponding point in T^1M has dense image (note that it isn't necessary). By this point, the reader should be convinced that the geodesic flow is an interesting dynamical system worthy of study.

A natural question to ask about group actions is whether they are transitive or not. In the case of the geodesic flow, that will not be the case because of dimensional reasons. In that case, one can ask for a more generalized version of transitivity, which is ergodicity. To define ergodicity, we need some additional structure on our dynamical system: a measure. On a measure space, a measure preserving transformation is said to be ergodic if the only sets left invariant by the transformation are full measure or zero measure. Another way to say the same thing is that a transformation T is ergodic iff the only L^2 functions left invariant by T are the constant functions.

Example 1. Consider the circle S^1 with the Lebesgue measure. The transformation that sends x to $x + \alpha$, where α is an irrational angle, is ergodic.

To even be able to ask whether the geodesic flow is ergodic or not, we need a measure on the unit tangent bundle that's preserved by the geodesic flow. A good choice of such a measure is the Liouville measure, which locally looks like the product of the Riemannian measure on the base volume form with the Lebesgue measure on the sphere that forms the fibre. The natural question to ask now is whether the geodesic flow on our manifold is ergodic or not. As we shall see, the answer is yes when the manifold is negatively curved.

2 Symmetric spaces

If we want to study the geodesic flow on an arbitrary Riemannian manifold, it might not be easy, since there is no nice coordinate system on an arbitrary manifold, nor a nice expression for the Riemannian exponential map. We thus focus our attention on a reasonably nice class of manifolds called symmetric spaces. These are manifolds such that at every point, there is a direction reversing isometry. What this means is that at any point p, and any direction v, the manifold looks the same in the direction $-v$. Many of the Riemannian manifolds we're familiar with are symmetric spaces.

Example 2. The simply connected spaces of curvature 1, -1, or 0 are symmetric spaces, where the direction reversing isometry at a point p is given by $\exp_p(v) \mapsto \exp_p(-v)$. In particular, the hyperbolic plane $\mathbb{H}^2$ is a symmetric space, and an example which we'll revisit later.

In the three examples of symmetric spaces we've seen, the spaces can also be realized as homogeneous spaces, i.e. $\mathbb{R}^n$ is $\mathrm{Aff}(n)/O(n)$, i.e. the space of affine transformations modded out by the orthogonal group. The easy way to see this is to note that the space of affine transformations acts transitively on $\mathbb{R}^n$, and the stabilizer of a point is the orthogonal group. Furthermore, the direction reversing isometry is an involution of the group $\mathrm{Aff}(n)$ that fixes $O(n)$. Similarly, in the case of $\mathbb{H}^n$, the space can be thought of as $SO(n,1)/SO(n)$. Another way the space $\mathbb{H}^2$ can be realized is the quotient $SL(2)/SO(2)$. We will use this version of $\mathbb{H}^2$ in our calculations later.

In fact, all symmetric spaces are of this form: they can be described as their isometry group modulo their isotropy group with an involution that keeps the isotropy group fixed (modulo certain hypotheses on the groups) (a proof of this may be found in [Hel79]). This also lets us construct symmetric spaces that aren't just the constant curvature ones. One example of such a space is the space $SL(n)/SO(n)$ with the direction reversing involution given by $A \mapsto (A^{-1})^T$, for $n > 2$. We'll see why this space doesn't have constant curvature later.

Recall that we're interested studying *compact* manifolds, but many of the symmetric spaces we've seen turn out to be non-compact. One way to get around this issue and get an even larger collection of spaces is to consider locally symmetric spaces, which are spaces obtained by quotienting out a symmetric space by a discrete fixed point free subgroup of the isometry group. If the discrete subgroup is large enough, the locally symmetric space we obtain will be compact (or have finite volume). To represent a locally symmetric space as a homogeneous space, let G/K be the symmetric space corresponding to the universal cover, and let Γ be the discrete subgroup of the isometry. Then the locally symmetric space is $\Gamma\backslash G/K$, i.e. right cosets of Γ in G/K. What this means is that any Riemannian manifold whose universal cover is a symmetric space turns out to be locally symmetric. In particular, this means all constant curvature complete manifolds are locally symmetric.

3 Translating geometry to algebra

As promised in the first section, we shall prove that the geodesic flow on compact negatively curved spaces is ergodic. Since we want to use tools from the theory of Lie groups (and their representations), it will be helpful to restrict our attention to negatively curved symmetric spaces. Also, since all manifolds of constant curvature are locally symmetric, this class will include all hyperbolic manifolds in particular.

The first question we need to ask ourselves is when is a symmetric space G/K negatively (or non-positively) curved? An easy observation we can make is the following: since G/K is simply connected and has non-positive curvature, it must necessarily be non-compact. Once we have that, the following theorem tells us what G and K must be.

Theorem 1 (Proposition 2.1.1 from [Ebe96])**.** *If G/K is a symmetric space of non-compact type, then G is a semisimple Lie group with trivial center, and K is a maximal compact subgroup.*

Note that the condition of G being semisimple isn't enough: all it ensures is that the symmetric space is non-positively curved. We're however interested in negatively curved spaces, i.e. the sectional curvature should be bounded away from 0 at every point and every 2-plane. To translate this into the algebra of $\mathfrak{g}$ (the Lie algebra of G), we need the notion of rank. To do this, recall that our symmetric spaces were specified as a semisimple group G, a maximal compact group K, and an involution, which we'll denote by σ. Since σ is an involution, it decomposes $\mathfrak{g}$ as $\mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the eigenspace corresponding to 1, and $\mathfrak{p}$ is the eigenspace corresponding to -1.

It's not too hard to see that $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup K, and $\mathfrak{p}$ is isomorphic to the tangent space of the symmetric space. Once we have this description of the tangent space of G/K, the Riemann curvature tensor can be written down in a particularly nice manner (the actual calculations can be found in [Mau04]).

$$R(x, y) = \mathrm{ad}([x, y])$$

This shows that if we have two vectors x and y in $\mathfrak{p}$ that commute, then the sectional curvature of that 2-plane will be 0. That means if we want our space to have strictly negative curvature, we must ensure that $\mathfrak{p}$ doesn't have any 2-dimensional abelian subalgebra. Such a symmetric space is said to have rank 1. A proof of the fact that rank 1 implies strictly negative curvature may be found chapter 6 of [Mau04].

Example 3. It might also be useful to verify the curvature calculations for $\mathbb{H}^2$. If we represent the space as $SL(2)/SO(2)$, the Lie algebra $\mathfrak{sl}(2)$ decomposes as $\mathfrak{k} = \mathfrak{so}(2)$, which is 1-dimensional, and spanned by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mathfrak{p}$, the tangent space, which is spanned by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If we now compute the sectional curvature using this formula, and the inner product on $\mathfrak{p}$ being the Killing form, then we'll get that the sectional curvature is -2, which is what we expected, i.e. negative.

One can also find symmetric spaces of higher rank, i.e. spaces such that the maximal abelian subalgebra of $\mathfrak{p}$ has dimension greater than 1.

Example 4. The tangent space $\mathfrak{p}$ for the space $SL(n)/SO(n)$ is the set of $n \times n$ symmetric traceless matrices. One can easily check that this space has an $n-1$ dimensional abelian subalgebra, and in fact, that is the maximal abelian subalgebra. Thus, $SL(n)/SO(n)$ has rank $n-1$. In particular, the spaces $SL(n)/SO(n)$ for $n > 2$ do not have constant curvature.

The next step is to try to understand what the geodesic flow on G/K is in purely group theoretic terms. To do that, we'll first need to express the unit tangent bundle $T^1(G/K)$ in terms of G. This is particularly easy to do in the rank 1 case: the unit tangent bundle is just a homogeneous G-space.

Lemma 2 (Lemma 4.25 from [BM00])**.** *If G/K is a rank 1 symmetric space, then the unit tangent bundle of G/K is the following coset space.*

$$T^1(G/K) \cong G/M$$

Here M is the centralizer of $\mathfrak{a}$, which is a maximal abelian subalgebra of $\mathfrak{p}$.

This lemma is useful because it lets us write down a particularly simple expression for the geodesic flow. The expression will be in terms of H, which is the norm 1 generator of $\mathfrak{a}$.

$$\phi : \mathbb{R} \times T^1(G/K) \to T^1(G/K)$$
$$\phi : (t, gM) \mapsto gM \cdot \exp(tH)$$

The representation of the unit tangent bundle and geodesic flow for locally symmetric spaces of rank 1 is analogous: the only difference is that we're looking at right cosets of some discrete subgroup Γ. The unit tangent bundle looks like the following.

$$T^1(\Gamma \backslash G/K) \cong \Gamma \backslash G/M$$

The geodesic flow looks like the following.

$$\phi : (t, \Gamma gM) \mapsto \Gamma gM \cdot \exp(tH)$$

Example 5. When G/K is $SL(2)/SO(2)$, a maximal abelian subalgebra is generated by $H = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. Note that we picked this matrix rather than $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ because we wanted a norm 1 vector. To determine what M must be in this case, recall that M is the centralizer of H in $SO(2)$. In particular, that means M consists of elements of $SO(2)$ that commute with matrices of the form $\begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$. It's easy to see that M must be the two element subgroup consisting of identity and the negative of the identity matrix.

The last piece of data we need to translate before we can prove the ergodicity is the measure on the unit tangent bundle. Luckily that turns out to have a nice translation too. The Liouville measure on the unit tangent bundle $T^1(G/K)$ translates to the Haar measure on G/M.

4 Proving ergodicity

Let's recall what exactly we'd set out to prove.

Theorem. *The geodesic flow on a finite volume negatively curved symmetric space is ergodic with respect to the Liouville measure.*

In the previous section, we translated the above theorem into more Lie theoretic terms.

Theorem. *Given a rank 1 locally symmetric space (of non-compact type) $\Gamma \backslash G/K$, the flow given by the following formula is ergodic with respect to the Haar measure.*

$$\phi : (t, \Gamma gM) \mapsto \Gamma gM \cdot \exp(tH)$$

Here H is the generator of the maximal abelian subalgebra of the tangent space, and M is the centralizer of H in K.

Our strategy to prove ergodicity will be to prove that any L^2 function on $\Gamma \backslash G/K$ that's invariant under the flow is also invariant under the action of G. Since G acts transitively on $\Gamma \backslash G/K$, this will prove that the flow invariant functions are constant, which is what we want.

To begin with, consider the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, the abelian subalgebra generated by H.

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\lambda \neq 0 \in \mathfrak{a}^*} \mathfrak{g}^\lambda$$

Here, the subspace $\mathfrak{g}^\lambda$ corresponds to all those $v \in \mathfrak{g}$ such that $[H, v] = \lambda v$. Furthermore, since M is the subgroup of G that commutes with $\exp(tH)$, that means we can further decompose $\mathfrak{g}^0$ as $\mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is the Lie algebra of M.

Now consider the action of G on $L^2(\Gamma\backslash G/M)$ given by $g \circ f(x) := f(xg)$. This action is continuous and unitary, since G acts by isometries. To show that any $\exp(tH)$ invariant function is also G invariant, it will suffice to show that it is $\exp(Y)$ invariant for each Y in $\mathfrak{g}^\lambda$. First of all, note that any function is $\exp(Y)$ invariant for $Y \in \mathfrak{m}$, since the space is $\Gamma\backslash G/M$. So we only need to prove the invariance for $Y \in \mathfrak{g}^\lambda$, for $\lambda \neq 0$. If $\lambda > 0$, we have the following identity.

$$\lim_{n\to\infty} \exp(-nH)\exp(Y)\exp(nH) = e$$

This is true because we can rewrite the above expression as the following.

$$\exp(\mathrm{Ad}(\exp(-nH))Y)$$

But that itself simplifies to $\exp(e^{-\lambda n}Y)$. For $\lambda < 0$, we get a similar identity, but for $n \to -\infty$. And now we can apply a result about unitary representations of Lie groups.

Lemma 3 (Mautner's lemma [Mau57])**.** *Let G be a Lie group, and (ϕ, H) be a continuous unitary representation of G on some Hilbert space H. Suppose there exist g and h in G satisfying the following identity.*

$$\lim_{n\to\infty} g^n h g^{-n} = e$$

Then any vector fixed by g is also fixed by h.

Proof. Suppose v is fixed by $\phi(g)$. We can estimate the difference between $\phi(h)v$ and v using the fact that ϕ is unitary.

$$\begin{aligned} |\phi(h)v - v| &= |\phi(h)\phi(g^{-n})v - \phi(g^{-n})v| \\ &= |\phi(g^n)\phi(h)\phi(g^{-n})v - v| \end{aligned}$$

If we now take the limit of the last expression as $n \to \infty$, we get that $\phi(h)v = v$, which is what we wanted. $\square$

We apply Mautner's lemma to the expression we had to conclude that any function invariant under $\exp(H)$ is also invariant under Y, and this proves the result.

References

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